Life is really Nonlinear.

Given $F: R^n \to R^n$, we seek to solve $F(x) = 0$

Assuming F is differentiable $\Rightarrow$ $\mathrm{F}'(x)_{ij} = \dfrac{\partial f_i}{\partial x_j}(x)$ exists.

From Calculus, one has:

Theorem 0:

$$F(x) - F(x^*) = \int_0^1 F'\left(x^* + t(x - x^*)\right)(x - x^*)\,dt \quad ----(1)$$

provided $x^* \in \Omega$ and x is sufficiently near x^*.

Definition 1:

$G: \Omega \subset R^N \to R^M$ is Lipschitz continuous on Ω.

if $\quad \|G(x) - G(y)\| \le \gamma \|x - y\|$ for some γ>0.

K is a contraction mapping on Ω if K is Lipschitz continuous and 0<γ<1.

Theorem 1:

If K is a contraction mapping with Lipschtz constant γ

then $\exists$ a unique fixed point x^* such that

$x_k = K(x_{k-1}) \to x^*$ q-linearly with q-factor γ (very slow)

Proof:

(1) $\|x_k - x_0\| \leq \left\| \sum_{i=1}^{k} x_i - x_{i-1} \right\| \leq \sum_{i=1}^{k} \gamma^{i-1} \|x_i - x_0\| \leq \frac{1}{1-\gamma} \|x_1 - x_0\|$

(2) $\|x_{n+k} - x_n\| \leq \gamma \|x_{n+k-1} - x_{n-1}\| \leq \gamma^n \|x_k - x_0\| \leq \frac{\gamma^n}{1-\gamma} \|x_1 - x_0\| \ \to 0 \ n \to \infty$

$\Rightarrow \{x_n\}$ Cauchy sequence $\Rightarrow x_n \to x^*$ for some x^*

Definition 2:

(i) $x_n \to x^*$ q-quadratically (fast)

if $\|x_{n+1} - x^*\| \leq K \|x_n - x^*\|^2$ for some $K>0$

(ii) $x_n \to x^*$ q-superlinearly with q order $\alpha>1$

if $\|x_{n+1} - x^*\| \leq K \|x_n - x^*\|^\alpha$ (faster than q-superlinearly)

(iii) $x_n \to x^*$ q-superlinearly

if $\lim_{n\to\infty} \frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|} = 0$

(iv) $x_n \to x^*$ q-linearly with q-factor $\alpha \in (0,1)$

if $\|x_{n+1} - x^*\| \leq \alpha \|x_n - x^*\|$ (slowest)

Application: Consider $y' = f(y)$ with $y(t_0) = y_0$

$$y' \approx \frac{y^{n+1} - y^n}{\Delta t}$$

The backward Euler method is

$$\frac{y^{n+1} - y^n}{\Delta t} = f\left(y^{n+1}\right) \Rightarrow y^{n+1} = y^n + f\left(y^{n+1}\right)\Delta t$$

To obtain y^{n+1}, one needs to solve

$$y = K(y) = y^n + f(y)\Delta t \quad ------(2)$$

If f is Lipschitz continuous with Lipschitz constant M,

$$\left(\frac{\left\|K(y) - K\left(y^*\right)\right\|}{\left\|y - y^*\right\|} \le \frac{\Delta t \left\| f\left(y^*\right) - f(y)\right\|}{\left\|y^* - y\right\|} \le \Delta t \cdot M \right)$$

the mapping K is a contraction when $\Delta t < \frac{1}{M}$.

By Theorem 1, (2) can be solved by the fixed point iteration when time step $\Delta t < \frac{1}{M}$.

Exercise:

Using the backward Euler method to solve

(i) $y' = \varepsilon y$ with $y(0) = 1$ $(\varepsilon = 1, 10, 100)$

(ii) $y' = \cos y$ with $y(0) = 0$

Standard assumption in Nonlinear iterations.

1. The equation has a solution x^*.
2. $F^{'}$ is Lipschtz continuous with Lipschtz constant γ.
3. $F^{'}\left(x^*\right)$ is non-singular.

Newton's method and stopping criterion of nonlinear iterations:

suppose x^* be the root of $F\left(x\right)$ and x is near x^* by Theorem 0,

we have $F\left(x\right)=\int_0^1 F^{'}\underbrace{\left(x^*+t\left(x-x^*\right)\right)}_{\tilde{x}\ \left(\tilde{x}\text{ is between }x^*\text{ and }x\right)}\left(x-x^*\right)dt \quad -(3)$

Since $F^{'}$ is Lipschitz $\dfrac{\left|F^{'}\left(\tilde{x}\right)-F^{'}\left(x^*\right)\right|}{\left|\tilde{x}-x^*\right|}<\gamma$

$\Rightarrow \left\|F^{'}\left(\tilde{x}\right)\right\|<\left\|F^{'}\left(x^*\right)\right\|+\gamma\left\|\tilde{x}-x^*\right\|$

When x is close enough to x^* $\left(\text{such that } \left\|\tilde{x}-x^*\right\|<\dfrac{1}{\gamma}\left\|F^{'}\left(x^*\right)\right\|\right)$

one has $\left\|F^{'}\left(\tilde{x}\right)\right\|<2\left\|F^{'}\left(x^*\right)\right\| \quad -(4)$

$\overset{(3)(4)}{\Rightarrow} \left\|F\left(x\right)\right\|\le 2\left\|F^{'}\left(x^*\right)\right\|\left\|x-x^*\right\|=2\left\|F^{'}\left(x^*\right)\right\|\left\|e\right\| \quad -(5)$

(here $e=x-x^*$)

Moreover, let $e = x - x^*$

consider $F'\left(x^*\right)^{-1} F(x) = F'\left(x^*\right)^{-1} \int_0^1 F'\left(x^* + t\left(x - x^*\right)\right) dt$

$$= e - \int_0^1 \left[I - F'\left(x^*\right)^{-1} \left(F'\left(x^* + t\left(x - x^*\right)\right)\right)\right] edt$$

$$\Rightarrow \left\| F'\left(x^*\right)^{-1} F(x) \right\| \geq \|e\| - \left\| \int_0^1 \left[I - F'\left(x^*\right)^{-1} \left(F'\left(x^* + t\left(x - x^*\right)\right)\right)\right] edt \right\|$$

$$\geq \|e\| - \int_0^1 \left\| I - F'\left(x^*\right)^{-1} F'\left(x^* + t\left(x - x^*\right)\right) \right\| \|e\| \quad \text{-----}(*)$$

Since $F'\left(x^*\right)$ is nonsigular, when x is close enough to x^*

$\left(\text{i.e. } x^* + t\left(x - x^*\right) \text{ is even closer to } x^*\right),$ one has

$$\left\| I - F'\left(x^*\right)^{-1} F'\left(x^* + t\left(x - x^*\right)\right) \right\| < \frac{1}{2} \qquad ----(6)$$

$$(*) \Rightarrow \left\| F'\left(x^*\right)^{-1} F(x) \right\| \geq \frac{1}{2} \|e\| \qquad ----(7)$$

Hence, from (5) and (7), we have

$$\frac{\|e\|}{2\left\| F'\left(x^*\right)^{-1} \right\|} \leq \|F(x)\| \leq 2\left\| F'\left(x^*\right) \right\| \|e\| \quad -(8)$$

$$\Rightarrow \begin{cases} \dfrac{\|F(x)\|}{\|F(x_0)\|} = 4 \underbrace{\left\| F'\left(x^*\right)^{-1} \right\| \left\| F'\left(x^*\right) \right\|}_{= K\left(F'\left(x^*\right)\right)} \cdot \dfrac{\|e\|}{\|e_0\|} \\ \underbrace{\dfrac{\|e\|}{\|e_0\|}}_{\text{relative error}} \leq 4K\left(F'\left(x^*\right)\right) \underbrace{\dfrac{\|F(x)\|}{\|F(x_0)\|}}_{\text{relative residual}} \end{cases} \quad -(9)$$

Exercise: Show $\exists \delta > 0$ such that

$$\left\| F'(x)^{-1} \right\| \leq 2 \left\| F'\left(x^*\right)^{-1} \right\| \quad \text{for } x \in B_\delta\left(x^*\right) \quad -(10)$$

(*) consider $A = F'\left(x^* + t\left(x - x^*\right)\right),\ B = F'\left(x^*\right)^{-1}$

$\overset{(6)}{\Rightarrow}$ we have $\|I - AB\| < \dfrac{1}{2}$ and AB is nonsingular.

$$A^{-1} = B\left(I - (I - AB)\right)^{-1}$$

$$\Rightarrow \left\|A^{-1}\right\| \leq \|B\| \left\| \left(I - (I - AB)\right)^{-1} \right\| \leq \frac{\|B\|}{1 - \|I - AB\|} \leq 2\|B\|$$

Theorem 2:

Let the standard assumptions hold, $\exists \delta$ such that if $x_0 \in B_\delta\left(x^*\right)$,

the newton iteration $x_{n+1} = x_n - F'\left(x_n\right)^{-1} F\left(x_n\right) \ \to x^*$ q-quadratically

Proof:

Consider $\underbrace{x^* - x_{n+1}}_{e_{n+1}} = \underbrace{x^* - x_n}_{e_n} - F'\left(x_n\right)^{-1} F\left(x_n\right)$

$$= F'\left(x_n\right)^{-1} \left(F'\left(x_n\right) e_n - F\left(x_n\right)\right)$$

$$\overset{\text{By Theorem 0}}{=} F'\left(x_n\right)^{-1} \int_0^1 \left(F'\left(x_n\right) - F'\left(x^* + t e_n\right)\right) e_n \, dt$$

$$\Rightarrow \ \left\|e_{n+1}\right\| \underset{F' \text{ is Lip}}{\leq} \left\|F'\left(x_n\right)^{-1}\right\| \left\| \int_0^1 \gamma \left\|(t-1) e_n\right\| \left\|e_n\right\| dt \right\| \underset{(10)}{\leq} K_n \left\|e_n\right\|^2$$

$\left(\text{here, } K_n = \left\|F'\left(x^*\right)^{-1}\right\| \gamma,\ \gamma = \text{Lipschtz constant}\right)$

$\Rightarrow \quad x_n \to x^*$ q-quadratically.

Remark:

1. $(9) \Rightarrow$ The stopping criterion should be determined according to the condition number of the Jacobian matrix $F\left(x^*\right)$ for the relative error to be less than a given tolerance.

2. Moreover, for the Newton's method, Theorem 2 implies that

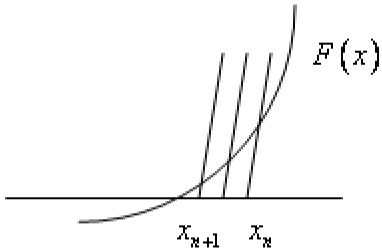
$$e_n = e_{n+1} + F'\left(x_n\right)^{-1} F\left(x_n\right)$$

$$\overset{\text{Theorem 2}}{\Rightarrow} \quad \left\|e_n\right\| = \underbrace{\mathrm{O}\left(\left\|e_n\right\|^2\right)}_{\left(\substack{\text{can be ignored when} \\ F'\left(x^*\right) \text{ is well-conditioned}}\right)} + \left\|F'\left(x_n\right) \backslash F\left(x_n\right)\right\|$$

For absolute error to be less than a given tolerance ε, the newton iteration should be stopped when $\left\|F'\left(x_n\right) \backslash F\left(x_n\right)\right\| < \varepsilon.$

3.Checking the quadratic convergence rate of the Newton method, we check $\dfrac{\left\|x_{n+1} - x_n\right\|}{\left\|x_n - x_{n-1}\right\|^2} \to$ a constant as $n \to \infty$ instead of checking $\dfrac{\left\|x - x_{n+1}\right\|}{\left\|x - x_n\right\|^2} \to$ a constant as $n \to \infty$

Chord method



$$x_{n+1} = x_n - F'(x_0)^{-1} F(x_n) \quad -(11)$$

If the standard assumption holds, the chord method converges q-linearly to the root x^*.

Remark 1:

$F'(x_0)^{-1}$ can be replaced by approximate inverse(preconditioner) B^{-1}(where $\|I - B^{-1}A\| < 1$ and B^{-1} is easier to compute).

$$\overset{(11)}{\Rightarrow} x_{n+1} = x_n - B^{-1}F(x_n) \quad -(12)$$

Remark 2:

When F' is difficult to compute, we can approximate F' by difference approximation.

$$(F')_j \approx (D_h F)_j = \begin{cases} \dfrac{F(x + h\|x\|e_j) - F(x)}{h \cdot \|x\|} & x \neq 0 \\ \dfrac{F(he_j) - F(0)}{h} & x = 0 \end{cases}$$

(In one-dimension case, this approach is called the secant method)

Algorithm of Chord method

1. $\gamma_0 = F(x_0)$.
2. compute $F'(x_0)$ and Factor $F'(x_0) = LU$.
3. do while $\|F(x)\| \geq \varepsilon \cdot F(x_0)$
   (a) Solve $LUs = -F(x)$
   (b) $x = x + s$
   (c) Evaluate $F(x)$

Theorem 3.

Let the standard assumption hold.

Then, $\exists \widetilde{K}$, δ and $\delta_1 > 0$ such that if $x_c \in B_\delta$ and $\|\Delta(x_c)\| < \delta_1$ then $x_+ = x_c - \left(F'(x_c) + \Delta(x_c)\right)^{-1}\left(F(x_c) + \varepsilon(x_c)\right)$ is defined and satisfies

$$\|x - x_+\| \leq \widetilde{K}\left(\|x - x_c\|^2 + \|\Delta(x_c)\|\|x - x_c\| + \|\varepsilon(x_c)\|\right)$$

Proof.

Let $x_+^N = x_c - F'(x_c)^{-1} F(x_c)$ be the Newton update.

$$\Rightarrow x_+ = x_+^N + \left(F'(x_c)^{-1} - \left(F'(x_c) + \Delta(x_c)\right)^{-1}\right)F(x_c) - \left(\left(F'(x_c) + \Delta(x_c)\right)^{-1}\right)\left(\varepsilon(x_c)\right)$$

$$\stackrel{(5)}{\Rightarrow} \|e_+\| \leq K\|e_c\|^2 + 2\left\|F'(x_c)^{-1} - \left(F'(x_c) + \Delta(x_c)\right)^{-1}\right\|\left\|F'(x^*)\right\|\|e_c\| + \left\|\left(F'(x_c) + \Delta(x_c)\right)^{-1}\right\|\|\varepsilon(x_c)\| \quad -(13)$$

If $\left\|\Delta\left(x_c\right)\right\| \le \dfrac{\left\|F'\left(x^*\right)^{-1}\right\|^{-1}}{4}$ then $\overset{(10)}{\Rightarrow}$ $\left\|\Delta x_c\right\| \le \dfrac{1}{2}\left\|F'\left(x_c\right)^{-1}\right\|^{-1}$

$$\Rightarrow \left\|\left(F'\left(x_c\right)+\Delta\left(x_c\right)\right)^{-1}\right\| = \left\|\left(F'\left(x_c\right)\left(I+F'\left(x_c\right)^{-1}\Delta\left(x_c\right)\right)\right)^{-1}\right\|$$

$$\le \left\|F'\left(x_c\right)^{-1}\right\|\left\|\left(I+F'\left(x_c\right)^{-1}\Delta\left(x_c\right)\right)^{-1}\right\|$$

$$\le \left\|F'\left(x_c\right)^{-1}\right\|\frac{1}{1-\left\|F'\left(x_c\right)^{-1}\right\|\left\|\Delta\left(x_c\right)\right\|}$$

$$\le 2\left\|F'\left(x_c\right)^{-1}\right\| \le 4\left\|F'\left(x^*\right)^{-1}\right\| \quad -(14)$$

Moreover, by the same argument, one can show

$$\left\|F'\left(x_c\right)^{-1}-\left(F'\left(x_c\right)+\Delta\left(x_c\right)\right)^{-1}\right\| \le 8\left\|F'\left(x_c\right)^{-1}\right\|^2\left\|\Delta x_c\right\| \quad -(15)$$

plug (14) (15) into (13), we have

$$\left\|e_+\right\| \le \widetilde{K}\left(\left\|e_c\right\|^2+\left\|\Delta x_c\right\|\left\|e_c\right\|+\left\|\varepsilon\left(x_c\right)\right\|\right),$$

here, $\widetilde{K} = K+16\left\|F'\left(x^*\right)^{-1}\right\|^2\left\|F'\left(x^*\right)\right\|+4\left\|F'\left(x^*\right)^{-1}\right\|$

Theorem 4.

Let standard assumption holds. There are K_c and $\delta > 0$ such that if $x_0 \in B\left(\delta\right)$, the chord iterates converge q-linearly to x^* and $\left\|e_{n+1}\right\| \le K_c\left\|e_0\right\|\left\|e_n\right\| \quad -(++)$

Proof.

Let δ be small enough so that Theorem 3 hold, and $\varepsilon(x_c)=0$,

$$\Delta(x_c)=F^{'}(x_0)-F^{'}(x_c) \quad \text{for } x_0\in B_\delta(x^*)$$

$$\Rightarrow \|\Delta(x_c)\|\le\gamma\|x_c-x_0\| \quad (F \text{ is Lipschtz.})$$

$$\le\gamma\left(\underbrace{\|x_c-x^*\|}_{e_c}+\underbrace{\|x_0-x^*\|}_{e_0}\right)$$

By Theorem 3,

$$\|e_{n+1}\|\le\widetilde{K}\left(\|e_n\|(1+\gamma)+\gamma\|e_0\|\right)\|e_n\|\le\left(\widetilde{K}(1+2\gamma)\delta\right)\|e_n\|$$

(assume $\|e_n\|<\|e_0\|$. This is true when δ is chosen such that $\|e_0\|<\delta$ and $K(1+2\gamma)\|e_0\|<1$.)

Clearly, the chord iterates converge q-linearly when $\widetilde{K}(1+2\gamma)\delta<1$, let $K_c=\widetilde{K}(1+2\gamma)$. The theorem is proved.

Theorem 5. Let the standard assumption hold.

Then $\exists K_B>0$, $\delta>0$ and $\delta_1>0$ such that if $x_0\in B(\delta)$ and approximate inverse $B(x)$ satisfies

$$\left\|I-B\cdot F^{'}(x^*)\right\|=\rho(x)<\delta_1$$

for all $x\in B(\delta)$, then the iteration

$$x_{n+1}=x_n-B(x_n)F(x_n)$$

converges q-linearly to x^* and

$$\|e_{n+1}\|\le K_B\left(\rho(x_n)+\|e_n\|\right)\|e_n\|.$$

(using chord iteration to accerlate iterations)

Shamanskii method:

Alternation of a Newton step with a sequence of chord steps.

$$(*)\begin{cases} y_1 = x_c - F'\left(x_c\right)^{-1} F\left(x_c\right) \\ y_{j+1} = y_j - F'\left(x_c\right)^{-1} F\left(y_j\right) \quad\quad 1 \le j \le m-1 \\ x_+ = y_m \end{cases}$$

go for next newton step.

When m=1, (*) ≡ Newton iteration

When m=∞, (*) ≡ Chord iteration

Algorithm of Shamanskii

1. $\gamma_0 = \left\| F\left(x\right) \right\|$
2. Do while $\left\| F\left(x\right) \right\| > \tau\gamma_0$
   (a) compute $F'\left(x\right)$
   (b) $F'\left(x\right) = LU$
   (c) for $j = 1 \sim m$
      (i) solve $LUs = -F\left(x\right)$
      (ii) $x = x + s$
      (iii) Evaluate $F\left(x\right)$
      (iv) if $F\left(x\right) \le \tau\gamma_0$, break
   (d) if $F\left(x\right) \le \tau\gamma_0$, break

Theorem 6.

Let m ≥ 1 be given, $\exists\, K_s > 0,\ \delta > 0$ such that if $x_0 \in B_\delta\left(x^*\right)$, the Shamanskii iterates converge with q-order $m+1$ and $\left\| e_{n+1} \right\| \le K_s \left\| e_n \right\|^{m+1}$

Remark 3:

(1) When $F^{'}$ is approximated by $\left(D_h F\right)_j$ the difference approximation,

$\Delta^h\left(x_c\right) = F^{'}\left(x_c\right) - \left(D_h F\right)_j F\left(x_c\right)$ with $\left\|\Delta^h\left(x_c\right)\right\| < \delta_1$

if Standard assumptions hold and $\left\|e_0\right\| < \delta\left(\text{with good initial guess}\right)$,

basically there is no diiference in using $\left(D_h F\right)_j$ and the exact $F^{'}\left(x_c\right)$

in the chord iterations. The convergence rate is at least q-linearly.

(2) When $\left\|e_n\right\| < \left\|\Delta^h\left(x_n\right)\right\|$, Theorem 3 impiles that no meaningful

error reduction can be obtain by iterations,

$$\left(\because \left\|e_{n+1}\right\| \le K\left(\left\|e_n\right\|^2 + \left\|\Delta^h\left(x_n\right)\right\|\left\|e_n\right\| + \left\|\varepsilon\left(x_n\right)\right\|\right)\right)$$

when the error $\left\|\varepsilon\left(x_n\right)\right\|$ in evaluation of the function value F

admits no further reduction.

Remark 4.

The conclusions in Remark 3 can be generalized to the approximate inverse B^{-1} when $\left\|B - F^{'}\left(x\right)\right\| \ll \delta_1$. This further implies that, in the approximate Newton step

$$x_{n+1} = x_n - B^{-1}F\left(x_n\right),$$

the system $B\left(\Delta x_n\right) = -F\left(x_n\right)$ needs not to be solve exactly.

Instead of solving $B\left(\Delta x_n\right)=-F_n$, one can solve $\frac{1}{2}$

$\widetilde{B}\left(\Delta x_n\right)=-F\left(x_n\right)$ as long as $\left\|\widetilde{B}-B\right\| \ll \delta_1$.

here $\widetilde{B}$ can be a preconditioner of B such as a

few steps of $\begin{cases} \text{stationary iterations} \\ \text{CG iterations} \\ \text{PCG iterations} \\ \text{GMRES iterations} \end{cases}$, etc.

The newton iterative method consists of solving $x_{n+1}=x_n-F^{'}\left(x_n\right)^{-1}F\left(x_n\right)$ by this approximation is called the inexact Newton Methods.

Please refer to C.T. kelly's "Iterative Methods" for detail error analysis of the inexact Newton method.

Broyden's Method

Broyden's method is locally superlinearly convergent! (in between Newton and chord method)

In one-dimension space, consider the secant method

$$x_{n+1}=x_n-\underbrace{\left(\frac{F\left(x_n\right)-F\left(x_{n-1}\right)}{x_n-x_{n-1}}\right)^{-1}}_{\theta^{-1}}F\left(x_n\right)$$

Remark 5: the secant iterates converge q-superlinearly.

Proof:

Let δ be small enough that theorem 3 holds and $\varepsilon(x_c)=0$

$$\|e_+\| \le \widetilde{K}\left(\|e_c\|^2 + \|\Delta(x_c)\|\|e_c\| + \varepsilon(x_c)\right) \quad -(16)$$

Since, by theorem 0, one has

$$F(x_c) - F(x_-) = \int_0^1 F'(x_- + t(x_c - x_-))(x_c - x_-)\,dt$$

$$\Rightarrow \frac{F(x_c)-F(x_-)}{x_c - x_-} - F'(x_c) = \int_0^1 F'(x_- + t(x_c - x_-)) - F'(x_c)\,dt$$

$$\Rightarrow \|\Delta(x_c)\| \le \int_0^1 \gamma(1-t)\|x_- - x_c\|\,dt = \frac{\gamma}{2}\|x_- - x_c\|$$

$$\le \frac{\gamma}{2}\left(\|e_c\| + \|e_-\|\right)$$

$$\overset{\text{plug into (16)}}{\Rightarrow} \quad \|e_+\| \le \widetilde{K}\left(\left(1+\frac{\gamma}{2}\right)\|e_c\|^2 + \frac{\gamma}{2}\|e_-\|\|e_c\|\right)$$

Choose δ such that $\widetilde{K}(1+\gamma)\delta < 1$ $\left(\Rightarrow \|e_n\| < \|e_{n-1}\| < \cdots < \|e_1\| < \|e_0\|\right)$

$e_+ = x_{n+1} - x = e_{n+1},\ e_c = x_n - x = e_n,\ e_- = x_{n-1} - x = e_{n-1}$

$$\Rightarrow \quad \frac{\|e_{n+1}\|}{\|e_n\|} \le \widetilde{K}\left(\left(1+\frac{\gamma}{2}\right)\|e_n\| + \frac{\gamma}{2}\|e_{n-1}\|\right)$$

$$\le \widetilde{K}(1+\gamma)\|e_{n-1}\| \quad \to 0 \text{ as } n \to \infty$$

Hence result.

Broyden method computes

$$B_+ = \underbrace{B_c}_{=\left(F'(x_n)\ or\ (D_h F)_j\right)} + \frac{(y - B_c s)s^T}{s^T s} = B_c + \underbrace{\frac{F(x_+)s^T}{s^T s}}_{\text{rank-one update}}$$

here $y = F(x_+) - F(x_c)$ and $s = x_+ - x_c$

In one-dimention, $\begin{pmatrix} \text{considering } B_c = 0 \\ x_+ = x_n,\ x_c = x_{n-1} \end{pmatrix}$, the iteration

$x_{n+1} = x_n - B_+^{-1} F(x_n)$ becomes $x_{n+1} = x_n - \left(\frac{F(x_n) - F(x_{n-1})}{x_n - x_{n-1}} \right) F(x_n)$

Clearly, the secant method is a special case of the Broyden method.

The Broyden iterations can be written as following

(1) $x_+ = x_c - B_c^{-1} F(x_c)$

(2) compute $s = x_+ - x_c$ and $w = F(x_+)$

(3) update $B_c = B_c + \frac{ws^{-1}}{s^T s}$ (by rank-one update)

$x_c = x_+$

(4) repeat (1) ~ (3) until converge.

Obviously, we would like to ask the following questions:

Q1: When will Broyden iterates converge?

Q2: How fast the Broyden converge?

Answer of Q1 and Q2:

Consider $B_n = F'(x^*) + E_n$

The Dennis-More condition is $\lim_{n\to\infty} \frac{\|E_n S_n\|}{\|S_n\|} = 0$

where $S_n = x_{n+1} - x_n$.

Theorem 7. Let the standard assumption hold.
Let B_n be a sequence of nonsingular matrix, let $x_0 \in R^N$ and $x_{n+1} = x_n - B_n^{-1} F(x_n)$. Assume $x_n \neq x^*$ for any n.
Then, $x_n \to x^*$ q-superlinearly if and only if $x_n \to x^*$ and the Dennis-More conditon holds.

Since $-F(x_n) = B_n s_n = F'(x^*) s_n + E_n s_n$

we have $E_n s_n = -F'(x^*) s_n - F(x_n)$, here $(s_n = x_{n+1} - x_n)$

$$\overset{(e_n = x_n - x^*)}{=} \quad -F'(x^*)(e_{n+1} - e_n) - F(x_n) \quad -(*)$$

By theorem 0,

$$F'(x^*) e_n - F(x_n) = \int_0^1 \left(F'(x^*) - F'(x^* + e_n)\right) e_n dt$$

$$\Rightarrow \; \left\| F'(x^*) e_n - F(x_n) \right\| \le \frac{\gamma}{2} \|e_n\|^2$$

$$\overset{(*)}{\Rightarrow} \; \|E_n s_n\| \le \left\| F'(x^*) e_{n+1} \right\| + \frac{\gamma}{2} \|e_n\|^2$$

So, if $x_n \to x^*$ q-superlinearly, given any $\varepsilon > 0$,

we have

$$\|e_{n+1}\| < \frac{1}{2} \|e_n\|, \; \|e_{n+1}\| < \frac{\varepsilon}{4} \left\| F'(x^*) \right\|^{-1} \|e_n\| \quad \text{and } \|e_n\| < \frac{1}{2\gamma} \varepsilon$$

for n large enough. Moreover, $\because \; \frac{1}{2} \|e_n\| \le \|e_n\| - \|e_{n+1}\| \le \|s_n\| \le 2 \|e_n\|$

$$\Rightarrow \; \frac{\|E_n s_n\|}{\|s_n\|} \le 2 \frac{\|E_n s_n\|}{\|e_n\|} \le 2\left(\frac{\varepsilon}{4} + \frac{\varepsilon}{4}\right) = \varepsilon.$$

By the definition of limit, one has

$$\lim_{n \to \infty} \frac{\|E_n s_n\|}{\|s_n\|} = 0 \qquad \Rightarrow \text{ The Dennis-Mor'e condition holds}$$

On the other hand,

if $\lim_{n \to \infty} \frac{\|E_n s_n\|}{\|s_n\|} = 0$ and $x_n \to x^*$

we want to show

$$\|e_{n+1}\| < \eta_n \|e_n\| \quad \text{and } \eta_n \to 0 \text{ as } n \to \infty.$$

From the Broyden iteration, one has

$$x^* - x_{n+1} = x^* - x_n + B_n^{-1} F(x_n)$$

$$= B_n^{-1}\left(B_n\left(x^* - x_n\right) + \int_0^1 F'\left(x_n + t\left(x_n - x^*\right)\right)\left(x_n - x^*\right) dt \right)$$

$$\Rightarrow e_{n+1} = B_n^{-1}\left[E_n e_n - \int_0^1 \left(F'\left(x^*\right) - F'\left(x_n - te_n\right)\right) e_n dt \right]$$

$$= B_n^{-1}\left[E_n\left(e_{n+1} + s_n\right) - \int_0^1 \left(F'\left(x^*\right) - F'\left(x_n - te_n\right)\right) e_n dt \right]$$

$$\Rightarrow \left(B_n - E_n\right) e_{n+1} = E_n s_n - \int_0^1 \left(F'\left(x^*\right) - F'\left(x_n - te_n\right)\right) e_n dt$$

$$\Rightarrow e_{n+1} = F'\left(x^*\right)^{-1}\left[E_n s_n - \int_0^1 \left(F'\left(x^*\right) - F'\left(x_n - te_n\right)\right) e_n dt \right]$$

$$\Rightarrow \left\|e_{n+1}\right\| \le \left\|F'\left(x^*\right)^{-1}\right\| \left[\left\|E_n s_n\right\| + \frac{\gamma}{2}\left\|e_n\right\|^2 \right]$$

$$\Rightarrow \left\|e_{n+1}\right\| \le \left\|F'\left(x^*\right)^{-1}\right\| \left[2\frac{\left\|E_n s_n\right\|}{\left\|s_n\right\|} + \frac{\gamma\left\|e_n\right\|}{2} \right] \left\|e_n\right\|.$$

Since $\lim_{n\to\infty} \frac{\left\|E_n s_n\right\|}{\left\|s_n\right\|} = 0$ and $\left\|e_n\right\| \to 0,$

we have $\eta_n = \left\|F'\left(x^*\right)^{-1}\right\| \left[2\frac{\left\|E_n s_n\right\|}{\left\|s_n\right\|} + \frac{\gamma\left\|e_n\right\|}{2} \right] \to 0$ as n $\to \infty$.

here, $x_n \to x^*$ q-superlinearly.

Theorem 7.1 Let the standard assumption holds.
Then there are δ and δ_B such that if $x_0 \in B_\delta$ and $\left\|E_0\right\|_2 < \delta_B$,
the Broyden sequence for the data $\left(F, x_0, B_0\right)$ exist
and $x_n \to x^*$ q-superlinearly.

Observation:

$$\omega = F\left(x_{n+1}\right),\ x_{n+1} = x_n - B_n^{-1}F\left(x_n\right)\ \Rightarrow B_n s_n = -F\left(x_n\right)$$

(&) $E_n = B_n - F^{'}\left(x^*\right)$, we have

$$E_{n+1} = B_{n+1} - F^{'}\left(x^*\right) = B_n + \frac{\omega s^T}{s^T s} - F^{'}\left(x^*\right) = E_n + \frac{\omega s_n^T}{\left\|s_n\right\|^2}$$

$$E_{n+1}^T = \left(I - \frac{s_n s_n^T}{\left\|s_n\right\|^2}\right)E_n^T + \frac{s_n s_n^T}{\left\|s_n\right\|^2}E_n^T + \frac{s_n \omega_n^T}{\left\|s_n\right\|^2}$$

$$= \left(I - \frac{s_n s_n^T}{\left\|s_n\right\|^2}\right)E_n + \frac{s_n}{\left\|s_n\right\|^2}\left(s_n^T B_n^T - s_n^T F^{'}\left(x^*\right) + F\left(x_{n+1}\right)^T\right)$$

$$= \left(I - \frac{s_n s_n^T}{\left\|s_n\right\|^2}\right)E_n + \frac{s_n}{\left\|s_n\right\|^2}\left(-F\left(x_n\right) - F^{'}\left(x^*\right)s_n + F\left(x_{n+1}\right)\right)^T$$

by (&) and theorem 0

$$= \left(I - \underbrace{\frac{s_n s_n^T}{\left\|s_n\right\|^2}}_{P}\right)E_n^T + \frac{s_n}{\left\|s_n\right\|^2}\left(\underbrace{\int_0^1\left(F^{'}\left(x_n + t\left(x_{n+1} - x_n\right)\right) - F^{'}\left(x^*\right)\right)dt}_{\Delta_n} \cdot s_n\right)^T$$

$$= \left(I - \frac{s_n s_n^T}{\left\|s_n\right\|^2}\right)E_n + \frac{s_n s_n^T}{\left\|s_n\right\|^2}\cdot\left(\Delta_n^T\right)$$

$$\Rightarrow E_{n+1}^T = \left(I - P\right)E_n^T + P\Delta_n^T$$

$$= E_n^T - \frac{s_n}{\left\|s\right\|}\left(\frac{s_n}{\left\|s_n\right\|}\right)^T \cdot E_n^T + P\Delta_n^T \quad --(**)$$

Lemma 1. Let $\{\theta_n\} \subset \{\hat{\theta}, 2-\hat{\theta}\}, \ \ 0<\hat{\theta}<1$

Let $\{\varepsilon_{\text{n}}\}_{n=0}^{\infty} \subset R^N$ be such that $\sum_{\text{n}} \|\varepsilon_n\|_2 < \infty$ and

$\{\eta_n\}_{n=0}^{\infty}$ be a set of vector that $\|\eta_n\|_2 = 1$ or 0.

If $\{\varphi_n\}_{n=1}^{\infty}$ is given by $\varphi_{\text{n+1}} = \varphi_n - \theta_n \left(\eta_n^T \varphi_n\right)\eta_n + \varepsilon_n$

then $\lim_{n\to\infty} \eta_n^T \varphi_n = 0.$

Consider $\varphi_{\text{n}} = E_n^T \phi, \ \eta_n = \dfrac{s_n}{\|s_n\|_2}, \ \varepsilon_{\text{n}} = P_n \Delta_n^T \phi.$

Apply Lemma 1 to (**) with $\theta_n = 1,$ we have

$$\lim_{n\to\infty} \frac{s_n^{\ T}}{\|s_n\|} \cdot E_n^{\ T} \phi = 0 \quad \text{for any } \phi$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{\left(E_n s_n\right)^T}{\|s_n\|} \cdot \phi = 0 \quad \text{for any } \phi$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{\|E_n s_n\|}{\|s_n\|} = 0 \qquad \text{the Dennis-Mor'e condition!}$$

$\Rightarrow \quad x_n \to x^*$ q-superlinearly by Theorem 7.

To prove Theorem 7.1, now the only thing we needs to do is to choose δ, δ_B, such that the assumption $\sum_{n=1}^{\infty} \|\varepsilon_n\| < \infty$ in Lemma 1 holds.

$$\sum_{n=1}^{\infty}\|\varepsilon_n\| = \sum_{n=1}^{\infty}\left\|P_n\Delta_n^T\phi\right\| \le \sum_{n=1}^{\infty}\int_0^1\left\|F^{'}\left(x_n+t\left(x_{n+1}-x_n\right)\right)-F^{'}\left(x^*\right)\right\|dt\|\phi\|$$

$$\le \sum_{n=1}^{\infty}\left(\frac{\gamma}{4}\right)\|e_{n+1}+e_n\| \le \sum_{n=1}^{\infty}\left(\frac{\gamma}{4}\right)\left(\|e_{n+1}\|+\|e_n\|\right) \quad -(***)$$

By theorem 3, there exist a constant K such that

$\|e_{n+1}\| < K\left(\|e_n\|+\|\Delta(x_n)\|\right)\|e_n\|,$ here $\Delta(x_n) = F'(x^*) - F'(x_n) + E_n.$

Clearly, when choosing $\delta_0, \delta_{\mathrm{B}}$ small enough such that

$\|e_n\| < \dfrac{1}{2K}$ and $\|\Delta(x_n)\| < \dfrac{1}{K},$ we have $\|e_{n+1}\| < \|e_n\|.$

As a result $\sum_{n=1}^{\infty}\|\varepsilon_n\| < \sum_{n=1}^{\infty}\left(\frac{\gamma}{2}\right)\|e_n\|.$

The series converges by the ratio test, because $\lim_{n\to\infty}\dfrac{\|e_n\|}{\|e_{n+1}\|} < 1.$

Finally, let's prove Lemma 1.

Observation: $\theta_{\mathrm{n}}\left(2-\theta_n\right) > \hat{\theta}^2 > 0$ for $\theta_{\mathrm{n}} \in (0,1)$

(1) consider $\varepsilon_n = 0$

$$\left\langle\varphi_{n+1},\varphi_{n+1}\right\rangle = \left\langle\varphi_n - \theta_n\left(\eta_n^T\varphi_n\right)\eta_n, \varphi_n - \theta_n\left(\eta_n^T\varphi_n\right)\eta_n\right\rangle$$

$$= \left\langle\varphi_n,\varphi_n\right\rangle - 2\theta_n\left(\eta_n^T\varphi_n\right)^2 + \theta_n^2\eta_n^2\left(\eta_n^T\varphi_n\right)^2$$

$$\|\eta_{\mathrm{n}}\| = 1 \text{ or } 0 \Rightarrow \|\varphi_{n+1}\|^2 \le \|\varphi_n\|^2 - \theta_n\left(2-\theta_n\right)\left(\eta_n^T\varphi_n\right)^2$$

$$\le \|\varphi_n\|^2 - \hat{\theta}^2\left(\eta_n^T\varphi_n\right)^2$$

For any $\mathrm{M} > 0,$

$$\sum_{\mathrm{n}=0}^{\mathrm{M}}\left(\eta_{\mathrm{n}}^{\mathrm{T}}\varphi_n\right)^2 < \left(\hat{\theta}^{-1}\right)^2\sum_{n=0}^{M}\|\varphi_n\|^2 - \|\varphi_{n+1}\|^2 = \hat{\theta}^{-2}\left(\|\varphi_0\|^2 - \|\varphi_{M+1}\|^2\right)$$

$$< \hat{\theta}^2\|\varphi_0\|^2 < \infty$$

$$\Rightarrow \lim_{n\to\infty}\eta_{\mathrm{n}}^{\mathrm{T}}\varphi_n = 0$$

(2) Consider $\varepsilon_{\mathrm{n}} \neq 0$:

let's use the inequality $\sqrt{a^2 - b^2} \leq a - \dfrac{b^2}{2a}$

$$\left\|\varphi_n - \theta_n\left(\eta_{\mathrm{n}}^{\mathrm{T}}\varphi_n\right)\eta_n\right\| \leq \sqrt{\left\|\varphi_n\right\|^2 - \theta_n\left(2-\theta_n\right)\left(\eta_{\mathrm{n}}^{\mathrm{T}}\varphi_n\right)^2}$$

$$\leq \left\|\varphi_{\mathrm{n}}\right\| - \frac{\theta_n\left(2-\theta_n\right)\left(\eta_{\mathrm{n}}^{\mathrm{T}}\varphi_n\right)^2}{2\left\|\varphi_n\right\|}$$

$$\Rightarrow \left\|\varphi_{n+1}\right\| \leq \left\|\varphi_n\right\| - \frac{\theta_n\left(2-\theta_n\right)\left(\eta_{\mathrm{n}}^{\mathrm{T}}\varphi_n\right)^2}{2\left\|\varphi_n\right\|} + \left\|\varepsilon_n\right\|$$

$$\Rightarrow \left(\eta_{\mathrm{n}}^{\mathrm{T}}\varphi_n\right)^2 \leq \frac{2\left\|\varphi_n\right\|}{\theta_n\left(2-\theta_n\right)}\left(\left\|\varphi_n\right\| - \left\|\varphi_{n+1}\right\| + \left\|\varepsilon_n\right\|\right)$$

Since $\left\|\varphi_{\mathrm{n+1}}\right\| \leq \left\|\varphi_n\right\| + \left\|\varepsilon_n\right\|$ and $\lim\limits_{\mathrm{n}\to\infty}\left\|\varepsilon_n\right\| = 0$

$\Rightarrow \exists\ \mu$ such that $\left\|\varphi_n\right\| < \mu$ for all n

$$\Rightarrow \sum_{\mathrm{n}=0}^{\mathrm{M}}\left(\eta_{\mathrm{n}}^{\mathrm{T}}\varphi_n\right)^2 \leq 2\hat{\theta}^{-2}\mu\left(\left\|\varphi_0\right\| - \left\|\varphi_{M+1}\right\| + \sum_{n=0}^{M}\left\|\varepsilon_n\right\|\right) < \infty \text{ for any given M}$$

$$\Rightarrow \lim_{n\to\infty}\eta_{\mathrm{n}}^{\mathrm{T}}\varphi_n = 0$$